

# Implied Volatility String Dynamics <sup>\*</sup>

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## Abstract

A primary goal in modelling the dynamics of implied volatility surfaces (IVS) aims at reducing complexity. For this purpose one fits the IVS each day and applies a principal component analysis using a functional norm. This approach, however, neglects the degenerated string structure of the implied volatility data and may result in a severe modelling bias. We propose a dynamic semiparametric factor model, which approximates the IVS in a finite dimensional function space. The key feature is that we only fit in the *local* neighborhood of the design points. Our approach is a combination of methods from functional principal component analysis and backfitting techniques for additive models. The model is found to have an approximate 10% better performance than the typical naïve trader models. The model can be a backbone in risk management serving for value at risk computations and scenario analysis.

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# 1 Introduction

Successful trading, hedging and risk managing of option portfolios crucially depends on the accuracy of the underlying pricing models. Departing from the pioneering foundations of option theory laid by Black and Scholes (1973), Merton (1973) and Harrison and Kreps (1979), new valuation approaches are continuously developed and existing models are refined. However, despite these pervasive developments, the model of Black and Scholes (1973) remains the pivot in modern financial theory and the benchmark for sophisticated models, be it from a theoretical or practical point of view.

The popularity of the Black and Scholes (BS) model is likely due to its clear and easy-to-communicate set of assumptions. Based on the geometric Brownian motion for the underlying asset price dynamics, and continuous trading in a complete and frictionless market, simple closed form solutions for plain vanilla calls and puts are derived: given the current underlying price  $S_t$  at time  $t$ , the option's strike price  $K$ , its expiry date  $T$  the prevailing riskless interest rate  $r_t$ , and an estimate of the (expected) market volatility  $\sigma$ , option prices are straightforward to compute.

The crucial parameter in option valuation by BS is the market volatility  $\sigma$ . Since it is unknown, one studies *implied* volatility  $\hat{\sigma}$ . Implied volatility is derived by inverting the BS formula for a cross section of options with different strikes and maturities traded at the same point in time. As is visible in the left panel of Figure 1 for May 2, 2000 (i.e. 20000502, a notation we will use from now on), implied volatilities display a remarkable curvature across the rescaled strike dimension moneyness  $\kappa$ , which is also called ‘smile’ or ‘smirk’, and – albeit to a lesser degree – a term structure across time to maturity  $\tau$ , which fluctuates through time  $t$ . This dependence, captured by the function  $\hat{\sigma}_t : (\kappa, \tau) \rightarrow \hat{\sigma}_t(\kappa, \tau)$ , is called

implied volatility surface (IVS). Apparently, it is in contrast with the BS framework in which volatility is assumed to be a constant across strikes, time to maturity and time.

There is a considerable amount of literature which aims at reconciling this empirical antagonism with financial theory. Generally, this may be achieved by including another degree of freedom into option pricing models. Well known examples are stochastic volatility models, (Hull and White; 1987; Stein and Stein; 1991; Heston; 1993), models with jump diffusions, Bates (1996), or models building on general Lévy processes, e.g. based on the inverse Gaussian, Barndorff-Nielsen (1997), and generalized hyperbolic distribution, Eberlein and Prause (2002). These approaches capture the smile and term structure phenomena and the complexity of its dynamics to some extent, Tompkins (2001). For the case of deterministic local volatility models Dumas et al. (1998) documented in an extensive study time inconsistency for a number of parametric volatility functions.

[Figure 1 about here.]

[Figure 2 about here.]

A new interpretation of the inaccuracy of the advanced models asserts that option markets due to their depth and liquidity behave more and more self-governed: apart from the underlying asset price dynamics, they are additionally driven by demand and supply conditions acting in options markets. In a comprehensive study Bakshi et al. (2000) show that a significant proportion of changes in option prices can neither be attributed to classical option pricing theory nor to market micro structure frictions. From this point of view, one interprets the IVS as a ‘state variable’, i.e. as an additional financial indicator, ruling in option markets, that reflects in an instantaneous and forward-looking manner the current state of



the market and the dynamics of its evolution. Thus, refined statistical model building of the IVS determines vitally the accuracy of applications in trading and risk-management.

In modelling the IVS one faces two main challenges. First, the data design is degenerated: due to trading conventions, observations of the IVS occur only for a small number of maturities such as 1, 2, 3, 6, 9, 12, 18, and 24 months to expiry on issuance. Consequently, implied volatilities appear in a row like pearls strung on a necklace, Figure 1, or, in short: as ‘strings’. This pattern is also visible in the right panel of Figure 1, which plots the data design as seen from top. Options belonging to the same string have a common time to maturity. As time passes, the strings move through the maturity axis towards expiry while changing levels and shape in a random fashion. Second, also in the moneyness dimension, the observation grid does not cover the desired estimation grid at any point in time with the same density. Thus, even when the data sets are huge, for a large number of cases implied volatility observations are missing for certain sub-regions of the desired estimation grid. This is particularly virulent when transaction based data is used.

For the semi- or nonparametric approximations to the IVS that recently have been promoted by Aït-Sahalia and Lo (1998); Rosenberg (2000); Aït-Sahalia et al. (2001b); Cont and da Fonseca (2002); Fengler et al. (2003); Fengler and Wang (2003), this design may pose difficulties. For illustration, consider in Figure 2 (left panel) the fit of a standard Nadaraya-Watson estimator. Bandwidths are  $h_1 = 0.03$  for the moneyness and  $h_2 = 0.04$  for the time to maturity dimension (measured in years). The fit appears very rough, and there are huge holes in the surface, since the bandwidths are too small to ‘bridge’ the gaps between the maturity strings. In order to remedy this deficiency one would need to strongly increase bandwidths which may induce a large bias. Moreover, since the design is time-varying, bandwidths would also need to be adjusted anew for each trading day, which complicates

daily applications. Parametric models, e.g. as in Shimko (1993), Ané and Geman (1999), and Tompkins (1999) among others, are less affected by these data limitations, but seem to offer too little functional flexibility to capture the salient features of IVS patterns. Thus, parametric estimates may as well be biased.

In this paper we provide a new principal component approach for modelling the IVS: the complex dynamic structure of the IVS is captured by a low dimensional semiparametric factor model with time-varying coefficients. The IVS is approximated by unknown basis functions moving in a finite dimensional function space. The dynamics can be understood by using vector autoregression (VAR) techniques on the time-varying coefficients. Contrary to earlier studies, we will use only finite dimensional fits to implied volatilities which are obtained in the *local* neighborhood of strikes and maturities, for which implied volatilities are recorded at the specific day. Also, surface estimation *and* dimension reduction is achieved in one single step, not in two as is currently standard. Our technology can be seen as a combination of functional principal component analysis, nonparametric curve estimation and backfitting for additive models.

Let us denote the (ln)-implied volatility by  $Y_{i,j}$ , where the index  $i$  is the number of the day ( $i = 1, \dots, I$ ), and  $j = 1, \dots, J_i$  is an intra-day numbering of the option traded on day  $i$ . The observations  $Y_{i,j}$  are regressed on two dimensional covariables  $X_{i,j}$  that contain moneyness  $\kappa_{i,j}$  and maturity  $\tau_{i,j}$ . Moneyness is defined as  $\kappa_{i,j} = K_{i,j}/F_{t,i,j}$ , i.e. strike  $K_{i,j}$  divided by the underlying futures price  $F_{t,i,j}$  at time  $t_{i,j}$ . We also considered the one-dimensional case in which  $X_{i,j} = \kappa_{i,j}$ . However, since modelling the entire surface is more interesting, we will present results for this case only. The semi-parametric dynamic factor model is:

$$m_0(X_{i,j}) + \sum_{l=1}^L \beta_{i,l} m_l(X_{i,j}) , \quad (1)$$

where  $m_l$  are smooth basis functions ( $l = 0, \dots, L$ ). The IVS is approximated by a weighted sum of smooth functions  $m_l$  with weights  $\beta_{i,l}$  depending on time  $i$ . The factor loading  $\beta_i \stackrel{\text{def}}{=} (\beta_{i,1}, \dots, \beta_{i,L})^\top$  forms an unobserved multivariate time series. By fitting model (1), to the implied volatility strings we obtain approximations  $\hat{\beta}_i$ . We argue that VAR estimation based on  $\hat{\beta}_i$  is asymptotically equivalent to estimation based on the unobserved  $\beta_i$ . A justification for this is given in Fengler et al. (2004) where the relations to Kalman filtering are discussed.

Lower dimensional approximations of the IVS based on principal components analysis (PCA) have been used in Avellaneda and Zhu (1997) and Fengler et al. (2002) in an application to the term structure of implied volatilities, and in Skiadopoulos et al. (1999) and Alexander (2001) in studies across strikes. Fengler et al. (2003) use a common principal components approach to study several maturity groups across the IVS simultaneously, while Cont and da Fonseca (2002) propose a functional PCA perspective for the IVS. All these approaches treat the IVS as a stationary process, and do not take particular care for the degenerated string structure which is apparent in Figure 1.

Our modelling approach is also different in the following respect: for instance, in Cont and da Fonseca (2002) the IVS is fitted on a grid for each day. Afterwards a PCA using a functional norm is applied to the surfaces. This treatment follows the usual functional PCA approach as described in Ramsay and Silverman (1997). In our approach the IVS is fitted each day at the observed design points  $X_{i,j}$ . This leads to a minimization with respect to functional norms that depend on time  $i$ . We loose a nice feature of usual functional PCA though: when fitting the data for  $L$  and  $L^* = L + 1$ , the linear space spanned by  $\hat{m}_0, \dots, \hat{m}_L$

may not be contained in the one spanned by  $\hat{m}_0^*, \dots, \hat{m}_L^*$ . On the other hand we only make use of values of the implied volatilities at regions where they are observed. This avoids bias effects caused by global daily fits used in standard functional PCA's.

The model can be employed in several respects: given the estimated functions  $\hat{m}_l$  and the time series  $\hat{\beta}$ , scenario simulations of potential IVS scenarios are straightforward. They can help to give a more accurate assessment of market risk than previous approaches. Worst case scenarios can be identified, which provide additional supervision tools to risk managers. For trading, the model may be used as a tool of short range IVS prediction. Even when it is difficult to predict the levels of the IVS due to the behavior of the level dynamics likely close to a unit root, certain specific distortions of the IVS may be predictable. To traders the model offers a unified tool for volatility hedging of complex option positions.

Since this model delivers a dynamic description of the IVS and certain IVS strings it may also serve as a way to model state price density (SPD) dynamics: according to Breeden and Litzenberger (1978) the SPD can be derived as the second derivative of the price function of plain vanilla options. Since the data is sparse in strikes, an approach due to Shimko (1993) converts option prices into implied volatilities, fits the smile on a dense grid, and computes the option prices via BS. The derivative, numerically much more stable than before, is computed with respect to the price function now obtained. Thus, IVS dynamics can be converted into SPD dynamics, and exploited for pricing and trading such as in Aït-Sahalia et al. (2001b) and Blaskowitz et al. (2003).

The paper is organized as follows: in the following section implied volatilities are described and the semiparametric factor model is introduced. In Section 3 the model is applied to DAX option implied volatilities for the sample period 1998 to May 2001. Further model

extensions and modifications are discussed in the conclusion.

## 2 Time-dependent implied volatility modelling

### 2.1 The semi-parametric factor model

Implied volatilities are derived from the BS option pricing formula for European calls and puts, Black and Scholes (1973). European style calls and puts are contingent claims on an asset  $S_t$  (paying no dividends for simplicity, here), which yield for strike  $K$  the pay-off  $\max(S_t - K, 0)$  and  $\max(K - S_t, 0)$ , respectively, at a given expiry day  $T$ . The asset price process  $S_t$  in the BS model is given by a geometric Brownian motion. The BS option pricing formula for calls is given by:

$$C_t^{BS}(S_t, K, \tau, r, \sigma) = S_t \Phi(d_1) - K e^{-r\tau} \Phi(d_2) , \quad (2)$$

where  $d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$  and  $d_2 = d_1 - \sigma\sqrt{\tau}$ .  $\Phi(u)$  denotes the cumulative distribution function of the standard normal distribution,  $\tau = T - t$  time to maturity of the option,  $r$  the riskless interest rate over the option's life time, and  $\sigma$  the diffusion coefficient of the Brownian motion. Put prices  $P_t$  are obtained via the put-call-parity  $C_t - P_t = S_t - e^{-r\tau}K$ .

The only unknown parameter in (2) is volatility  $\sigma$ . Given observed market prices  $\tilde{C}_t$ , implied volatility  $\hat{\sigma}$  is defined by:

$$C_t^{BS}(S_t, K, \tau, r, \hat{\sigma}) - \tilde{C}_t = 0 . \quad (3)$$

Due to monotonicity of the BS price in the volatility parameter  $\sigma$  there exists a unique solution  $\hat{\sigma} > 0$ . Define the moneyness metric  $\kappa_t = K/F_t$ , where  $F_t$  denotes the future price time  $t$ .

In the dynamic factor model, we regress  $Y_{i,j} \stackrel{\text{def}}{=} \ln\{\hat{\sigma}_{i,j}(\kappa, \tau)\}$  on  $X_{i,j} = (\kappa_{i,j}, \tau_{i,j})$  via non-parametric methods. We work with log-implied volatility data, since the data appears less skewed and potential outliers are scaled down after taking logs. This is common practice in the IVS literature, see e.g. Avellaneda and Zhu (1997); Cont and da Fonseca (2002). In order to estimate the nonparametric components  $m_l$  and the state variables  $\beta_{i,l}$  in (1), we borrow ideas from fitting additive models as in Stone (1986); Hastie and Tibshirani (1990); Horowitz et al. (2002). Our research is related to functional coefficient models such as Cai et al. (2000). Other semi- and nonparametric factor models include Connor and Linton (2000); Gouriéroux and Jasiak (2001); Fan et al. (2003); Linton et al. (2003b). Nonparametric techniques are now broadly used in option pricing, e.g. Broadie et al. (2000); Aït-Sahalia et al. (2001a); Aït-Sahalia and Duarte (2003), and interest rate modelling, e.g. Aït-Sahalia (1996); Ghysels and Ng (1989); Linton et al. (2003a).

Estimates  $\hat{m}_l$ ,  $(l = 0, \dots, L)$  and  $\hat{\beta}_{i,l}$   $(i = 1, \dots, I; l = 1, \dots, L)$  are defined as minimizers of the following least squares criterion ( $\hat{\beta}_{i,0} \stackrel{\text{def}}{=} 1$ ):

$$\sum_{i=1}^I \sum_{j=1}^{J_i} \int \left\{ Y_{i,j} - \sum_{l=0}^L \hat{\beta}_{i,l} \hat{m}_l(u) \right\}^2 K_h(u - X_{i,j}) du. \quad (4)$$

Here,  $K_h$  denotes a two-dimensional product kernel,  $K_h(u) = k_{h_1}(u_1) \times k_{h_2}(u_2)$ ,  $h = (h_1, h_2)$  based on a one-dimensional kernel  $k_h(v) = h^{-1}k(h^{-1}v)$ .

In (4) the minimization runs over all functions  $\hat{m}_l : \mathbb{R}^2 \rightarrow \mathbb{R}$  and all values  $\hat{\beta}_{i,l} \in \mathbb{R}$ . For illustration let us consider the case  $L = 0$  : implied volatilities  $Y_{i,j}$  are approximated by a surface  $\hat{m}_0$  that does not depend on time  $i$ . In this degenerated case,  $\hat{m}_0(u) = \sum_{i,j} K_h(u - X_{i,j})Y_{i,j} / \sum_{i,j} K_h(u - X_{i,j})$ , which is the Nadaraya-Watson estimate based on the pooled sample of all days. Note that in the algorithmic implementation of (4) the integral is replaced by Riemann sums on a fine grid.

Using (4) the implied volatility surfaces are approximated by surfaces moving in an  $L$ -dimensional affine function space  $\{\widehat{m}_0 + \sum_{l=1}^L \alpha_l \widehat{m}_l : \alpha_1, \dots, \alpha_L \in \mathbb{R}\}$ . The estimates  $\widehat{m}_l$  are not uniquely defined: they can be replaced by functions that span the same affine space. In order to respond to this problem, we select  $\widehat{m}_l$  such that they are orthogonal.

Replacing in (4)  $\widehat{m}_l$  by  $\widehat{m}_l + \delta g$  with arbitrary functions  $g$  and taking derivatives with respect to  $\delta$  yields for  $0 \leq l' \leq L$

$$\sum_{i=1}^I \sum_{j=1}^{J_i} \left\{ Y_{i,j} - \sum_{l=0}^L \widehat{\beta}_{i,l} \widehat{m}_l(u) \right\} \widehat{\beta}_{i,l'} K_h(u - X_{i,j}) = 0. \quad (5)$$

Furthermore, by replacing  $\widehat{\beta}_{i,l}$  by  $\widehat{\beta}_{i,l} + \delta$  in (4) and again taking derivatives with respect to  $\delta$ , we get for  $1 \leq l' \leq L$  and  $1 \leq i \leq I$

$$\sum_{j=1}^{J_i} \int \left\{ Y_{i,j} - \sum_{l=0}^L \widehat{\beta}_{i,l} \widehat{m}_l(u) \right\} \widehat{m}_{l'}(u) K_h(u - X_{i,j}) du = 0. \quad (6)$$

Introducing the following notation for  $1 \leq i \leq I$

$$\widehat{p}_i(u) = \frac{1}{J_i} \sum_{j=1}^{J_i} K_h(u - X_{i,j}), \quad (7)$$

$$\widehat{q}_i(u) = \frac{1}{J_i} \sum_{j=1}^{J_i} K_h(u - X_{i,j}) Y_{i,j}, \quad (8)$$

we obtain from (5)-(6) for  $1 \leq l' \leq L, 1 \leq i \leq I$ :

$$\sum_{i=1}^I J_i \widehat{\beta}_{i,l'} \widehat{q}_i(u) = \sum_{i=1}^I J_i \sum_{l=0}^L \widehat{\beta}_{i,l} \widehat{\beta}_{i,l} \widehat{p}_i(u) \widehat{m}_l(u), \quad (9)$$

$$\int \widehat{q}_i(u) \widehat{m}_{l'}(u) du = \sum_{l=0}^L \widehat{\beta}_{i,l} \int \widehat{p}_i(u) \widehat{m}_{l'}(u) \widehat{m}_l(u) du. \quad (10)$$

We calculate the estimates by iterative use of (9) and (10). We start by initial values  $\widehat{\beta}_{i,l}^{(0)}$  for  $\widehat{\beta}_{i,l}$ . A possible choice of the initial  $\widehat{\beta}$  could correspond to fits of IVS's that are piecewise

constant on time intervals  $I_1, \dots, I_L$ . This means for  $l = 1, \dots, L$  put  $\widehat{\beta}_{i,l}^{(0)} = 1$  (for  $i \in I_l$ ), and  $\widehat{\beta}_{i,l}^{(0)} = 0$  (for  $i \notin I_l$ ). Here  $I_1, \dots, I_L$  are pairwise disjoint subsets of  $\{1, \dots, I\}$  and  $\bigcup_{l=1}^L I_l$  is a strict subset of  $\{1, \dots, I\}$ . For  $r \geq 0$  we put  $\widehat{\beta}_{i,0}^{(r)} = 1$ . Define the matrix  $B^{(r)}(u)$  by its elements:

$$(B^{(r)}(u))_{l,l'} \stackrel{\text{def}}{=} \sum_{i=1}^I J_i \widehat{\beta}_{i,l'}^{(r-1)} \widehat{\beta}_{i,l}^{(r-1)} \widehat{p}_i(u), \quad 0 \leq l, l' \leq L, \quad (11)$$

and introduce a vector  $Q^{(r)}(u)$  with elements

$$Q^{(r)}(u)_l \stackrel{\text{def}}{=} \sum_{i=1}^I J_i \widehat{\beta}_{i,l}^{(r-1)} \widehat{q}_i(u), \quad 0 \leq l \leq L. \quad (12)$$

In the  $r$ -th iteration the estimate  $\widehat{m} = (\widehat{m}_0, \dots, \widehat{m}_L)^\top$  is given by

$$\widehat{m}^{(r)}(u) = B^{(r)}(u)^{-1} Q^{(r)}(u). \quad (13)$$

This update step is motivated by (9). The values of  $\widehat{\beta}$  are updated in the  $r$ -th cycle as follows: define the matrix  $M^{(r)}(i)$

$$(M^{(r)}(i))_{l,l'} \stackrel{\text{def}}{=} \int \widehat{p}_i(u) \widehat{m}_{l'}^{(r)}(u) \widehat{m}_l^{(r)}(u) du, \quad 1 \leq l, l' \leq L, \quad (14)$$

and define a vector  $S^{(r)}(i)$

$$S^{(r)}(i)_l \stackrel{\text{def}}{=} \int \widehat{q}_i(u) \widehat{m}_l(u) du - \int \widehat{p}_i(u) \widehat{m}_0^{(r)}(u) \widehat{m}_l^{(r)}(u) du, \quad 1 \leq l \leq L. \quad (15)$$

Motivated by (10), put

$$\left( \widehat{\beta}_{i,1}^{(r)}, \dots, \widehat{\beta}_{i,L}^{(r)} \right)^\top = M^{(r)}(i)^{-1} S^{(r)}(i). \quad (16)$$

The algorithm is run until only minor changes occur. In the implementation, we choose a grid of points and calculate  $\widehat{m}_l$  at these points. In the calculation of  $M^{(r)}(i)$  and  $S^{(r)}(i)$ , we replace the integral by a Riemann integral approximation using the values of the integrated functions at the grid points.



As discussed above,  $\widehat{m}_l$  and  $\widehat{\beta}_{i,l}$  are not uniquely defined. Therefore, we orthogonalize  $\widehat{m}_0, \dots, \widehat{m}_L$  in  $L^2(\widehat{p})$ , where  $\widehat{p}(u) = I^{-1} \sum_{i=1}^I \widehat{p}_i(u)$ , such that  $\sum_{i=1}^I \widehat{\beta}_{i,1}^2$  is maximal, and given  $\widehat{\beta}_{i,1}, \widehat{m}_0, \widehat{m}_1$ ,  $\sum_{i=1}^I \widehat{\beta}_{i,2}^2$  is maximal, and so forth. These aims can be achieved by the following two steps: first replace

$$\begin{aligned} \widehat{m}_0 & \quad \text{by} \quad \widehat{m}_0^{\text{new}} = \widehat{m}_0 - \gamma^\top \Gamma^{-1} \widehat{m}, \\ \widehat{m} & \quad \text{by} \quad \widehat{m}^{\text{new}} = \Gamma^{-1/2} \widehat{m}, \\ \begin{pmatrix} \widehat{\beta}_{i,1} \\ \vdots \\ \widehat{\beta}_{i,L} \end{pmatrix} & \quad \text{by} \quad \begin{pmatrix} \widehat{\beta}_{i,1}^{\text{new}} \\ \vdots \\ \widehat{\beta}_{i,L}^{\text{new}} \end{pmatrix} = \Gamma^{1/2} \left\{ \begin{pmatrix} \widehat{\beta}_{i,1} \\ \vdots \\ \widehat{\beta}_{i,L} \end{pmatrix} + \Gamma^{-1} \gamma \right\}, \end{aligned} \tag{17}$$

where  $\widehat{m} = (\widehat{m}_1, \dots, \widehat{m}_L)^\top$  and the  $(L \times L)$  matrix  $\Gamma = \int \widehat{m}(u) \widehat{m}(u)^\top \widehat{p}(u) du$ , or for clarity,  $\Gamma = (\Gamma_{l,l'})$ , with  $\Gamma_{l,l'} = \int \widehat{m}_l(u) \widehat{m}_{l'}(u) \widehat{p}(u) du$ . Finally, we have  $\gamma = (\gamma_l)$ , with  $\gamma_l = \int \widehat{m}_0(u) \widehat{m}_l(u) \widehat{p}(u) du$ .

Note that by applying (17),  $\widehat{m}_0$  is replaced by a function that minimizes  $\int \widehat{m}_0^2(u) \widehat{p}(u) du$ . This is evident because  $\widehat{m}_0$  is orthogonal to the linear space spanned by  $\widehat{m}_1, \dots, \widehat{m}_L$ . By the second equation of (17),  $\widehat{m}_1, \dots, \widehat{m}_L$  are replaced by orthonormal functions in  $L^2(\widehat{p})$ .

In a second step, we proceed as in PCA and define a matrix  $B$  with  $B_{l,l'} = \sum_{i=1}^I \widehat{\beta}_{i,l} \widehat{\beta}_{i,l'}$  and calculate the eigenvalues of  $B$ ,  $\lambda_1 > \dots > \lambda_L$ , and the corresponding eigenvectors  $z_1, \dots, z_L$ . Put  $Z = (z_1, \dots, z_L)$ . Replace

$$\widehat{m} \quad \text{by} \quad \widehat{m}^{\text{new}} = Z^\top \widehat{m}, \tag{18}$$

(i.e.  $\widehat{m}_l^{\text{new}} = z_l^\top \widehat{m}$ ), and

$$\begin{pmatrix} \widehat{\beta}_{i,1} \\ \vdots \\ \widehat{\beta}_{i,L} \end{pmatrix} \text{ by } \begin{pmatrix} \widehat{\beta}_{i,1}^{\text{new}} \\ \vdots \\ \widehat{\beta}_{i,L}^{\text{new}} \end{pmatrix} = Z^\top \begin{pmatrix} \widehat{\beta}_{i,1} \\ \vdots \\ \widehat{\beta}_{i,L} \end{pmatrix}. \quad (19)$$

After application of (18) and (19) the orthonormal basis  $\widehat{m}_1, \dots, \widehat{m}_L$  is chosen such that  $\sum_{i=1}^I \widehat{\beta}_{i,1}^2$  is maximal, and— given  $\widehat{\beta}_{i,1}, \widehat{m}_0, \widehat{m}_1$ — the quantity  $\sum_{i=1}^I \widehat{\beta}_{i,2}^2$  is maximal,  $\dots$ , i.e.  $\widehat{m}_1$  is chosen such that as much as possible is explained by  $\widehat{\beta}_{i,1} \widehat{m}_1$ . Next  $\widehat{m}_2$  is chosen to achieve maximum explanation by  $\widehat{\beta}_{i,1} \widehat{m}_1 + \widehat{\beta}_{i,2} \widehat{m}_2$ , and so forth.

The functions  $\widehat{m}_l$  are not eigenfunctions of an operator as in usual functional PCA. This is because we use a different norm, namely  $\int f^2(u) \widehat{p}_i(u) du$ , for each day. Through the norming procedure the functions are chosen as eigenfunctions in an  $L$ -dimensional approximating linear space. The  $L$ -dimensional approximating spaces are not necessarily nested for increasing  $L$ . For this reason the estimates cannot be calculated by an iterative procedure that starts by fitting a model with one component, and that uses the old  $L - 1$  components in the iteration step from  $L - 1$  to  $L$  to fit the next component. The calculation of  $\widehat{m}_0, \dots, \widehat{m}_L$  has to be fully redone for different choices of  $L$ .

## 2.2 Choice of $L$ , $h$ , and speed of convergence

For the choice of  $L$ , we consider the residual sum of squares for different  $L$ :

$$RV(L) \stackrel{\text{def}}{=} \frac{\sum_i \sum_j^{J_i} \left\{ Y_{i,j} - \sum_{l=0}^L \widehat{\beta}_{i,l} \widehat{m}_l(X_{i,j}) \right\}^2}{\sum_i \sum_j^{J_i} (Y_{i,j} - \bar{Y})^2}, \quad (20)$$

where  $\bar{Y}$  denotes the overall mean of the observations. The quantity  $1 - RV(L)$  is the portion of variance explained in the approximation, and  $L$  can be increased until a sufficiently high

level of fitting accuracy is achieved. This is a common selection method also in ordinary PCA.

For a data-driven choice of bandwidths, we propose an approach based on a weighted Akaike Information Criterion (AIC). We argue for using a weighted criterion, since the distribution of observations is very unequal, as was seen from Figure 5. To our experience, this leads to nonconvexity in the criterion and typically to inacceptably small bandwidths, see Fengler et al. (2003) for a first description of this problem. Given the unequal distribution of observations, it is natural to punish the criterion in areas where the distribution is sparse. For a given weight function  $w$ , consider:

$$\Delta(m_0, \dots, m_L) \stackrel{\text{def}}{=} E \frac{1}{N} \sum_{i,j} \{Y_{i,j} - \sum_{l=0}^L \beta_{i,l} m_l(X_{i,j})\}^2 w(X_{i,j}) , \quad (21)$$

for functions  $m_0, \dots, m_L$ . The expectation operator is denoted by  $E$ . We choose bandwidths such that  $\Delta(\hat{m}_0, \dots, \hat{m}_L)$  is minimal. According to the AIC this is asymptotically equivalent to minimizing:

$$\Xi_{AIC_1} \stackrel{\text{def}}{=} \frac{1}{N} \sum_{i,j} \{Y_{i,j} - \sum_{l=0}^L \hat{\beta}_{i,l} \hat{m}_l(X_{i,j})\}^2 w(X_{i,j}) \exp \left\{ 2 \frac{L}{N} K_h(0) \int w(u) du \right\} . \quad (22)$$

Alternatively, one may consider the computationally more easy criterion:

$$\Xi_{AIC_2} \stackrel{\text{def}}{=} \frac{1}{N} \sum_{i,j} \{Y_{i,j} - \sum_{l=0}^L \hat{\beta}_{i,l} \hat{m}_l(X_{i,j})\}^2 \exp \left\{ 2 \frac{L}{N} K_h(0) \frac{\int w(u) du}{\int w(u) p(u) du} \right\} . \quad (23)$$

Putting  $w(u) \stackrel{\text{def}}{=} 1$ , delivers the common AIC. This, however, does not take into account the quality of the estimation at the boundary regions or in regions where data is sparse, since in these regions  $p(u)$  is small. We propose to choose

$$w(u) \stackrel{\text{def}}{=} \frac{1}{p(u)} , \quad (24)$$

which gives equal weight everywhere as can be seen by the following considerations:

$$\begin{aligned}
\Delta(m_0, \dots, m_L) &= E \frac{1}{N} \sum_{i,j} \varepsilon^2 w(X_{i,j}) \\
&\quad + E \frac{1}{N} \sum_{i,j} \left[ \sum_{l=0}^L \beta_{i,l} \{m_l(X_{i,j}) - \widehat{m}_l(X_{i,j})\} \right]^2 w(X_{i,j}) \\
&\approx \sigma^2 \int w(u) p(u) du \\
&\quad + \frac{1}{N} \sum_{i,j} \int \left[ \sum_{l=0}^L \beta_{i,l} \{m_l(u) - \widehat{m}_l(u)\} \right]^2 w(u) p(u) du .
\end{aligned} \tag{25}$$

The two criteria become:

$$\Xi_{AIC_1} \stackrel{\text{def}}{=} \frac{1}{N} \sum_{i,j} \{Y_{i,j} - \sum_{l=0}^L \widehat{\beta}_{i,l} \widehat{m}_l(X_{i,j})\}^2 \widehat{p}(X_{i,j}) \exp \left\{ 2 \frac{L}{N} K_h(0) \int \frac{1}{\widehat{p}(u)} du \right\} , \tag{26}$$

and

$$\Xi_{AIC_2} \stackrel{\text{def}}{=} \frac{1}{N} \sum_{i,j} \{Y_{i,j} - \sum_{l=0}^L \widehat{\beta}_{i,l} \widehat{m}_l(X_{i,j})\}^2 \exp \left\{ 2 \frac{L}{N} K_h(0) \mu_\lambda^{-1} \int \frac{1}{\widehat{p}(u)} du \right\} , \tag{27}$$

where  $\mu_\lambda$  denotes the Lebesgue measure of the design set.

Under some regularity conditions, the AIC is an asymptotically unbiased estimate of the mean averaged square error (MASE). In our setting it would be consistent if the density of  $X_{i,j}$  did not depend on day  $i$ . Due to the irregular design, this is an unrealistic assumption. For this reason,  $\Xi_{AIC_1}$  and  $\Xi_{AIC_2}$  estimate a weighted versions of MASE.

In our AIC the penalty term does not punish for the number parameters  $\widehat{\beta}_{i,l}$  that are employed to model the time series. This can be neglected because we will use a finite dimensional model for the dynamics of  $\beta_{i,l}$ . The corresponding penalty term is negligible compared to the smoothing penalty term. A corrected penalty term that takes care of the parametric model of  $\widehat{\beta}_{i,j}$  will be considered in Section 3.3 where the prediction performance is assessed.

Clearly the choice of  $h$  and  $L$  are not independent. From this point of view, one may think about minimizing (26) or (27) over both parameters. However, our practical experience shows that for a given  $L$ , changes in the criteria from variation in  $h$  is small, compared to variation in  $L$  for a given  $h$ . To reduce the computational burden, we use (20) to determine model size  $L$ , and then (26) and (27) to optimize  $h$  for a given  $L$ .

Convergence of the iterations is measured by

$$Q_k(r) \stackrel{\text{def}}{=} \sum_{i=1}^I \int \left| \sum_{l=0}^L \widehat{\beta}_i^{(r)} \widehat{m}_l^{(r)}(u) - \widehat{\beta}_i^{(r-1)} \widehat{m}_l^{(r-1)}(u) \right|^k du . \quad (28)$$

As above  $(r)$  denotes the result from the  $r$ th cycle of the estimation. Here, we approximate the integral by a simple sum over the estimation grid. Putting  $k = 1, 2$ , we have an  $L^1$  and an  $L^2$  measure of convergence. Iterations are stopped when  $Q_k(r) \leq \epsilon_k$  for some small  $\epsilon > 0$ .

## 3 Applications

### 3.1 Data description and preparation

The data set under consideration contains tick statistics on the DAX future and DAX index options and was provided by the German-Swiss Futures Exchange EUREX for the period from January 1998 to May 2001. Both future and option data are contract based data, i.e. each single contract is registered together with its price, contract size, and time of settlement up to a hundredth second. Interest rate data in daily frequency, i.e. 1, 3, 6, 12 months FIBOR rates for the years 1998–1999 and EURIBOR rates for the period 2000–2001, were obtained from *Thomson Financial Datastream*. Interest rate data was linearly interpolated to approximate a ‘riskless’ interest rate for the option specific time to maturity.

In order to avoid German tax bias, option raw data has undergone a preparation scheme which is due to Hafner and Wallmeier (2001) and described in the following. The entire data set is stored in the Financial Database MD\*base, [www.mdtech.de](http://www.mdtech.de), maintained at CASE – Center for Applied Statistics and Economics, Berlin.

In a first step, we recover the DAX index values. To this end, we group to each option price observation  $H_t$  the future price  $F_t$  of the nearest available future contract, which was traded within a one minute interval around the observed option. The future price observation was taken from the most heavily traded futures contract on the particular day, usually the 3 months contract. The no-arbitrage price of the underlying index in a frictionless market without dividends is given by  $S_t = F_t e^{-r_{T_F,t}(T_F-t)}$ , where  $S_t$  and  $F_t$  denote the index and the future price respectively,  $T_F$  the future’s maturity date, and  $r_{T,t}$  the interest rate with maturity  $T - t$ .

The DAX index is a capital weighted performance index (Deutsche Börse; 2002), i.e. dividends less corporate tax are reinvested into the index. Therefore, at a first glance, dividend payments should have no or almost little impact on index options. However, when only the interest rate discounted future is used to recover implied volatilities from an inversion of the BS formula, implied volatilities of calls and puts can differ significantly. This discrepancy is especially large during spring when most of the 30 companies listed in the DAX distribute dividends. The point is best visible in Figure 3 from 20000404: implied volatilities of calls (crosses) and puts (circles) fall apart, thus violating the put-call-parity and general market efficiency considerations.

[Figure 3 about here.]

[Figure 4 about here.]

Hafner and Wallmeier (2001) argue that the marginal investor's individual tax scheme is different from the one actually assumed to compute the DAX index. Consequently, the net dividend for this investor can be higher or lower than the one used for the index computation. This discrepancy, which the authors call 'difference dividend', has the same impact as a dividend payment for an unprotected option, i.e. it drives a wedge into the option prices and hence into implied volatilities. Denote by  $\Delta D_{t,T}$  the time  $T$  value of this difference dividend incurred between  $t$  and  $T$ . Consider the dividend adjusted futures pricing formula:

$$F_t = S_t e^{r_F(T_F-t)} - \Delta D_{t,T_F}, \quad (29)$$

and the dividend adjusted put-call parity:

$$C_t - P_t = S_t - \Delta D_{t,T_H} e^{-r_H(T_H-t)} - K e^{-r_H(T_H-t)} \quad (30)$$

with  $T_H$  denoting the call's  $C_t$  and the put's  $P_t$  maturity date. Inserting equation (29) into (30) yields

$$C_t - P_t = F_t e^{-r_F(T_F-t)} + \Delta D_{t,T_H,T_F} - K e^{-r_H(T_H-t)}, \quad (31)$$

where  $\Delta D_{t,T_H,T_F} \stackrel{\text{def}}{=} \Delta D_{t,T_F} e^{-r_F(T_F-t)} - \Delta D_{t,T_H} e^{-r_H(T_H-t)}$  is the desired difference dividend.

The 'adjusted' index level

$$\tilde{S}_t = F_t e^{-r_F(T_F-t)} + \Delta D_{t,T_H,T_F} \quad (32)$$

is that index level, which ties put and call implied volatilities exactly to the same levels when used in the inversion of the BS formula.

For an estimate of  $\Delta \hat{D}_{t,T_H,T_F}$ , pairs of puts and calls of the strikes and same maturity are identified provided they were traded within a five minutes interval. For each pair the

$\Delta D_{t,T_H,T_F}$  is derived from equation (31). To ensure robustness  $\Delta \hat{D}_{t,T_H,T_F}$  is estimated by the median of all  $\Delta D_{t,T_H,T_F}$  of the pairs for a given maturity at day  $t$ . Implied volatilities are recovered by inverting the BS formula using the corrected index value  $\tilde{S}_t = F_t e^{-r_F(T_F-t)} + \Delta \hat{D}_{t,T_H,T_F}$ . Note that  $\Delta D_{t,T_H,T_F} = 0$ , when  $T_H = T_F$ . Indeed, when calculated also in this case,  $\Delta \hat{D}_{t,T_H,T_F}$  proved to be very small (compared with the index value), which supports the validity of this approach. The described procedure is applied on a daily basis throughout the entire data set from 199801 to 200105.

In Figure 4, also from 20000404, we present the data after correcting the discounted future price with an implied difference dividend  $\Delta \hat{D}_t = (10.3, 5.0, 1.9)^\top$ , where the first entry refers to 16 days, the second to 45 days and the third to 73 days to maturity. Implied volatilities of puts and calls converge to one single string, while the concavity of the put volatility smile is remedied, too. Note that the overall level of implied volatility string is *not* altered through that procedure.

Since the data is transaction based and may contain potential misprints or outliers, a filter is applied: observations with implied volatility less than 4% and bigger than 80% are dropped. Furthermore, we disregard all observations having a maturity  $\tau \leq 10$  days. After this filtering, the entire number of observations is more than 4.48 million contracts, i.e. is around 5 200 observations per day. All computations have been made with XploRe, [www.i-xplo.de](http://www.i-xplo.de).

Table 1 gives a short summary of our IVS data. Most heavy trading occurs in short term contracts, as is seen from the difference between median and mean of the term structure distribution of observations as well as from its skewness. Median time to maturity is 30 days (0.083 years). Across moneyness the distribution is slightly negatively skewed. Mean implied volatility over the sample period is 27.9%.



[Table 1 about here.]

### 3.2 Implied Volatility Surface Modelling

Implied volatilities are observed only at particular strings, but it is common practice to think about them as being the observed structures of an entire surface, the IVS. This becomes evident, when one likes to price and hedge options with intermediate maturities. We model ln-implied volatility on  $X_{i,j} = (\kappa_{i,j}, \tau_{i,j})^\top$ . The grid covers in moneyness  $\kappa \in [0.80, 1.20]$  and in time to maturity  $\tau \in [0.05, 0.5]$  measured in years.

In this model, we employ  $L = 3$  basis functions, which capture around 96.0% of the variations in the IVS. We believe this to be of sufficiently high accuracy. Bandwidths used are  $h_1 = 0.03$  for moneyness and  $h_2 = 0.04$  for time to maturity. This choice is justified by Table 4 which presents estimates for the two AIC criteria. Both criterion functions become very flat near the minimum, especially  $\Xi_{AIC_1}$ . However,  $\Xi_{AIC_2}$  assumes its global minimum in the neighborhood of  $h^* = (0.03, 0.04)^\top$ , which is why we opt for these bandwidths. In Table 4 we also display a measure of how factor loadings and basis functions change relative to the optimal bandwidth  $h^*$ . We compute:

$$V_{\hat{\beta}}(h_k) = \sqrt{\sum_{l=0}^L \text{var}\{|\hat{\beta}_{i,l}(h_k) - \hat{\beta}_{i,l}(h^*)|\}} , \quad (33)$$

$$\text{and } V_{\hat{m}}(h_k) = \sqrt{\sum_{l=0}^L \text{var}\{|\hat{m}_l(u; h_k) - \hat{m}_l(u; h^*)|\}} , \quad (34)$$

where  $h_k$  runs over the values given in Table 4, and  $\text{var}(x)$  denotes the variance of  $x$ . It is seen that changes in  $\hat{m}$  are 10 to 100 times higher in magnitude than those for  $\hat{\beta}$ . This corroborates the approximation in (25) that treats the factor loadings as known.

[Table 2 about here.]

In being able to choose such small bandwidths, the strength of our modelling approach is demonstrated: the bandwidth in the time to maturity dimension is so small that in the fit of a particular day, data from contracts with two adjacent time to maturities do not enter together  $\hat{p}_i(u)$  in (7) and  $\hat{q}_i(u)$  in (8). In fact, for given  $u'$ , the quantities  $\hat{p}_i(u')$  and  $\hat{q}_i(u')$  are zero most of the time, and only assume positive values for dates  $i$  when observations are in the local neighborhood of  $u'$ . The same applies to the moneyness dimension. Of course, during the entire observation period  $I$ , it is mandatory that observations for each  $u$  for some dates  $i$  are made.

[Figure 5 about here.]

In Figure 5 we display the  $L^1$  and  $L^2$  measures of convergence. Convergence is achieved quickly. The iterations were stopped after 25 cycles, when the  $L^2$  was less than  $10^{-5}$ . Figures 6 to 8 display the functions  $\hat{m}_1$  to  $\hat{m}_4$  together with contour plots. We do not display the invariant function  $\hat{m}_0$ , since it essentially is the zero function of the affine space fitted by the data: both mean and median are zero up to  $10^{-2}$  in magnitude. We believe this to be estimation error. The remaining functions exhibit a more interesting patterns:  $\hat{m}_1$  in Figure 6 is positive throughout, and mildly concave. There is little variability across the term structure. Since this function belongs to the weights with highest variance, we interpret it as the time dependent mean of the (log)-IVS, i.e. a *shift effect*.

[Figure 6 about here.]

[Figure 7 about here.]

[Figure 8 about here.]

Function  $\hat{m}_2$ , depicted in Figure 7, changes sign around the at-the-money region, which implies that the smile deformation of the IVS is exacerbated or mitigated by this eigenfunction. Hence we consider this function as a *moneyness slope effect* of the IVS. Finally,  $\hat{m}_3$  is positive for the very short term contracts, and negative for contracts with maturity longer than 0.1 years, Figure 8. Thus, a positive weight in  $\hat{\beta}_3$  lowers short term implied volatilities and increases long term implied volatilities:  $\hat{m}_3$  generates the term structure dynamics of the IVS, it provides a *term structure slope effect*.

To appreciate the power of the semiparametric dynamic factor model, we inspect again the situation of 20000502. In Figure 9 we compare a Nadaraya-Watson estimator (left panel) with the semi-parametric factor model (right panel). In the first case, the bandwidths are increased to  $h = (0.06, 0.25)^\top$  in order to remove all holes and excessive variation in the fit, while for the latter the bandwidths are kept at  $h = (0.03, 0.04)^\top$ . While both fits look quite similar at a first glance, the differences are best visible when both cases are contrasted for each time to maturity string separately, Figures 10 to 13. Generally, the standard Nadaraya-Watson fit exhibits a strong directional bias, especially in the wings of the IVS. For instance, for the short maturity contracts, Figure 10, the estimated IVS is too low both in the OTM put and the OTM call region. At the same time, levels are too high for the 45 days to expiry contracts, Figure 11. For the 80 days to expiry case, Figure 12, the fit exhibits an *S*-formed shape, although the data lie almost on a linear line. Also the semi-parametric factor model is not entirely free of a directional bias, but clearly the fit is superior.

[Figure 9 about here.]

[Figure 10 about here.]

[Figure 11 about here.]

[Figure 12 about here.]

[Figure 13 about here.]

[Figure 14 about here.]

[Table 3 about here.]

[Table 4 about here.]

Figure 14 shows the entire time series of  $\hat{\beta}_1$  to  $\hat{\beta}_3$ , summary statistics are given in Table 3 and contemporaneous correlation in Table 4. The correlograms given in the lower panel of Figure 14 display rich autoregressive dynamics of the factor loadings. The ADF tests at the 5% level, Table 5, indicate a unit root for  $\hat{\beta}_1$  and  $\hat{\beta}_2$ . Thus, one may model first differences of the first two loading series together with the levels of  $\hat{\beta}_3$  in a parsimonious VAR framework. Alternatively, since the results are only marginally significant, one may estimate the levels of the loading series in a rich VAR model. We opt for the latter choice and use a VAR(2) model, Table 6. The estimation also includes a constant and two dummy variables, assuming the value one right at those days and one day after, when the corresponding IV observations of the minimum time to maturity string (10 days to expiry) were to be excluded from the estimation of the semiparametric factor model, confer Section 3.1. This is to capture possible seasonality effects introduced from the data filter.

Estimation results are displayed in Table 6. In the equations of  $\widehat{\beta}_1$  and  $\widehat{\beta}_2$  the constants and dummies are weakly significant. However, for sake of clarity, estimation results on the constant and the dummy variables are not shown. As is seen all factor loadings follow AR(2) processes. There is also a number of remarkable cross dynamics: first order lags in the level dynamics,  $\widehat{\beta}_1$ , have a positive impact on the term structure,  $\widehat{\beta}_3$ . Second order lags in the term structure dynamics themselves influence positively the moneyiness slope effect,  $\widehat{\beta}_2$ , and negatively the shift variable  $\widehat{\beta}_1$ : thus shocks in the the term structure may decrease the level of the smile and aggravate the skew. Similar interpretations can be revealed from other significant coefficients in Table 6.

In earlier specifications of the model, we also included contemporaneous and lagged DAX returns into the regression equation. The results revealed a leverage interpretation of the first loading series, Black (1976): shocks in the underlying are negatively correlated with level changes in (implied) volatility, possibly due to a change in the debt-equity ratio. However, since for our horse-race in Section 3.3, we use a simple VAR without exogenous variables due to fairness, we do not show these results.

[Table 5 about here.]

[Table 6 about here.]

### 3.3 Prediction Performance

We now study the prediction performance of our model compared with a benchmark model. Model comparisons that have been conducted, for instance by Bakshi et al. (1997); Dumas et

al. (1998); Bates (2000); Jackwerth and Rubinstein (2001), often show that so called ‘naïve trader models’ perform best or only little worse than more sophisticated models. These models used by professionals simply assert that today’s implied volatility is tomorrow’s implied volatility. There are two versions: the sticky strike assumption pretends that implied volatility is constant at fixed strikes. The sticky delta or sticky moneyness version claims this for implied volatilities observed at a fixed moneyness or option delta, Derman (1999). We use the sticky moneyness model as our benchmark. There are two reasons for this choice: first, from a methodological point of view, as has been shown by Balland (2002) and Daglish et al. (2003), the sticky strike rule as an assumption on the stochastic process governing implied volatilities, is not consistent with the existence of a smile. The sticky moneyness rule, however, can be. Second, practically, since we estimate our model in terms of moneyness, sticky moneyness rule is most natural.

Our methodology in comparing prediction performance is as follows: as presented in Section 3.2, the resulting times series of latent factors  $\hat{\beta}_{i,l}$  is replaced by a times series model with fitted values  $\tilde{\beta}_{i,l}(\hat{\theta})$  based on  $\hat{\beta}_{i',l}$  with  $i' \leq i - 1$ ,  $1 \leq l \leq L$ , where  $\hat{\theta}$  is a vector of estimated coefficients seen in Table 6. Similarly to Section 2, we employ a AIC based on these *fitted* values as an asymptotically unbiased estimate of mean square prediction error.

For the model comparison we use criterion  $\Xi_{AIC_1}$  with  $w(u) \stackrel{\text{def}}{=} 1$ . Additionally we penalize the dimension of the fitted time series model  $\tilde{\beta}(\theta)$ :

$$\begin{aligned} \tilde{\Xi}_{AIC} \stackrel{\text{def}}{=} N^{-1} \sum_i^I \sum_j^{J_i} \left\{ Y_{i,j} - \sum_{l=0}^L \tilde{\beta}_{i,l}(\hat{\theta}) \hat{m}_l(X_{i,j}) \right\}^2 \\ \times \exp \left\{ 2 \frac{L}{N} K_h(0) \mu_\lambda + \frac{2 \dim(\theta)}{N} \right\} . \end{aligned} \quad (35)$$

In our case  $\dim(\theta) = 27$ , since we have for three equations six VAR-coefficients plus constant

and two dummy variables.

Criterion (35) is compared with the squared one-day prediction error of the sticky moneyness (StM) model:

$$\Xi_{StM} \stackrel{\text{def}}{=} N^{-1} \sum_i^I \sum_j^{J_i} (Y_{i,j} - Y_{i-1,j'})^2 . \quad (36)$$

In practice, since one hardly observes  $Y_{i,j}$  at the same moneyness as in  $i - 1$ ,  $Y_{i-1,j'}$  is obtained via a localized interpolation of the previous day's smile. Time to maturity effects are neglected, and observations, the previous values of which are lost due to expiry, are deleted from the sample.

Running the model comparison shows:

$$\begin{aligned} \Xi_{StM} &= 0.00476 \\ \tilde{\Xi}_{AIC} &= 0.00439 \end{aligned}$$

Thus, the model comparison reveals that the semi-parametric factor model is approximate 10% better than the naïve trader model. This is a substantial improvement given the high variance in implied volatility and financial data in general. An alternative approach would investigate the hedging performance of our model compared with other models, e.g. in following Engle and Rosenberg (2000). This is left for further research.

## 4 Conclusion and model extensions

In this study we present a modelling approach to the implied volatility surface (IVS) that takes care of the particular discrete string structure of implied volatility data. The technique

comes from functional PCA and backfitting in additive models. Unlike other studies, our ansatz is tailored to the degenerated design of implied volatility data by fitting basis functions in the local neighborhood of the design points only. We thus avoid bias effects.

Using transactions based DAX index implied volatility data from 1998 to May 2001, we recover a number of basis functions generating the dynamics of a single implied volatility string and surface. The functions may be intuitively interpreted as level, term structure and moneyness slope effects, and a term structure twist effect, known from earlier literature. We study the time series properties of parameters weights and complete the modelling approach by fitting vector autoregressive models. A three-dimensional VAR(2) describes IVS dynamics best.

Our modelling approach allows for a number of extensions that can be discussed. A first topic is efficiency. After the calculation of  $\hat{m}_0, \dots, \hat{m}_L$  and  $\hat{\beta}_{i,1}, \dots, \hat{\beta}_{i,L}$  the estimates of  $\beta_{i,l}$  may be replaced by the minimizers  $\tilde{\beta}_{i,1}, \dots, \tilde{\beta}_{i,L}$  of

$$\sum_{i=1}^I \sum_{j=1}^{J_i} \left\{ Y_{i,j} - \sum_{l=0}^L \tilde{\beta}_{i,l} \hat{m}_l(X_{i,j}) \right\}^2.$$

Discussions of other semiparametric models suggest that this estimate is more efficient than the estimate above. However, this approach has the disadvantage that it requires calculation of the components at all design points  $X_{i,j}$ .

Another extension would be to introduce into equation (4) weights  $w_{i,j}$ . For instance, in applications other than ours, the weights could correct for unequal values of  $J_i$  or could take care of heteroscedastic errors, e.g. variances depending on maturity. Alternatively weights may be derived from economic variables such the underlying trading volume associated with a particular implied volatility observation.



A third modelling approach would be to replace (4) by

$$\sum_{i=1}^I \sum_{j=1}^{J_i} \sum_{k=1}^I \int \left\{ Y_{i,j} - \widehat{m}_0(u) - \sum_{l=1}^L \widehat{\beta}_{i,l} \widehat{m}_{l,k}(u) \right\}^2 K_h(u - X_{i,j}) G(k - i) du,$$

where  $G$  is a smooth function. In this case a system of principal components is fitted to the data that *smoothly* develops in time. In trading applications a model must be continuously updated and recalibrated. Thus in the trading context, this third model extension may be a fruitful starting point for even better exploiting the transaction based tick data.

Finally, as mentioned earlier, the semi-parametric model has a natural relation with Kalman filtering. Kalman filtering is a way to recursively find solutions to discrete-data linear filtering problems, Kalman (1960). For, writing our model more compactly as:

$$\Theta(L)\beta_i = u_i \tag{37}$$

$$Y_{i,j} = m_0(X_{i,j}) + \sum_{l=1}^L \beta_{i,l} m_l(X_{i,j}) + \varepsilon_{i,j}, \tag{38}$$

where  $\Theta(L) \stackrel{\text{def}}{=} 1 - \theta_1 L - \theta_2 L^2 - \dots - \theta_p L^p$  denotes a polynomial of order  $p$  in the lag operator  $L$ , we receive the typical state-space representation, where  $u_i$  and  $\varepsilon_{i,j}$  is noise. Equation (38) is called the *state equation* depending on a parameter vector  $\theta$ . It relates the (unobservable) state  $i$  of the system to the previous step  $i - 1$ . The *measurement equation* (38) relates the state to the measurement, the IVS in our case. The difference to our work is that the time series modelling of  $\beta_i$  is done *after* recovering it. For the integrated approach see Fengler et al. (2004).

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		Min.	Max.	Mean	Median	Stdd.	Skewn.	Kurt.
All	Time to maturity	0.028	2.014	0.131	0.083	0.148	3.723	23.373
	Moneyiness	0.325	1.856	0.985	0.993	0.098	-0.256	5.884
	Implied volatility	0.041	0.799	0.279	0.256	0.090	1.542	6.000
1998	Time to maturity	0.028	2.014	0.134	0.081	0.148	3.548	22.957
	Moneyiness	0.386	1.856	0.984	0.992	0.108	-0.030	5.344
	Implied volatility	0.041	0.799	0.335	0.306	0.114	0.970	3.471
1999	Time to maturity	0.028	1.994	0.126	0.083	0.139	4.331	32.578
	Moneyiness	0.371	1.516	0.979	0.992	0.099	-0.595	5.563
	Implied volatility	0.047	0.798	0.273	0.259	0.076	0.942	4.075
2000	Time to maturity	0.028	1.994	0.130	0.083	0.151	3.858	23.393
	Moneyiness	0.325	1.611	0.985	0.992	0.092	-0.337	6.197
	Implied volatility	0.041	0.798	0.254	0.242	0.060	1.463	7.313
2001	Time to maturity	0.028	0.978	0.142	0.083	0.159	2.699	10.443
	Moneyiness	0.583	1.811	1.001	1.001	0.085	0.519	6.762
	Implied volatility	0.043	0.789	0.230	0.221	0.049	1.558	7.733

Table 1: *Summary statistics on the data base from 199801 to 200105 for the IVS application in Section 3.2, entirely and on an annual basis. 2001 is from 200101 to 200105, only.*



$h_1$	$h_2$	$\Xi_{AIC_1}$	$\Xi_{AIC_2}$	$V_{\hat{\beta}}$	$V_{\hat{m}}$
0.01	0.02	0.000737	0.00151	0.015	0.938
0.01	0.04	0.000741	0.00150	0.003	0.579
0.01	0.06	0.000739	0.00152	0.005	0.416
0.01	0.08	0.000736	0.00163	0.011	0.434
0.02	0.02	0.001895	0.00237	0.104	3.098
0.02	0.04	0.000738	0.00150	0.001	0.181
0.02	0.06	0.000741	0.00151	0.004	0.196
0.02	0.08	0.000742	0.00156	0.008	0.279
0.02	0.10	0.000744	0.00162	0.011	0.339
0.03	0.02	0.002139	0.00256	0.111	3.050
<b>0.03</b>	<b>0.04</b>	<b>0.000739</b>	<b>0.00149</b>	—	—
0.03	0.06	0.000743	0.00152	0.004	0.180
0.03	0.08	0.000743	0.00156	0.008	0.273
0.03	0.10	0.000744	0.00162	0.011	0.337
0.04	0.02	0.002955	0.00323	0.138	3.017
0.04	0.04	0.000743	0.00151	0.001	0.088
0.04	0.06	0.000746	0.00154	0.005	0.211
0.04	0.08	0.000745	0.00157	0.008	0.293
0.04	0.10	0.000746	0.00163	0.012	0.353
0.05	0.02	0.003117	0.00341	0.142	2.962
0.05	0.04	0.000748	0.00155	0.001	0.148
0.05	0.06	0.000749	0.00157	0.005	0.241
0.05	0.08	0.000748	0.00160	0.008	0.312
0.05	0.10	0.000749	0.00167	0.012	0.368
0.06	0.02	0.003054	0.00343	0.139	2.923
0.06	0.04	0.000755	0.00160	0.002	0.193
0.06	0.06	0.000756	0.00163	0.005	0.268
0.06	0.08	0.000754	0.00166	0.009	0.330
0.06	0.10	0.000754	0.00172	0.012	0.383

Table 2: *Bandwidth selection via AIC as given in (26) and (27) for different choices of  $h$ :  $h_1$  refers to moneyness and  $h_2$  to time to maturity measured in years; the bandwidth chosen is highlighted in bold. In all cases  $L = 3$ .  $V_{\hat{\beta}}$  and  $V_{\hat{m}}$  measure the change in  $\hat{\beta}$  and  $\hat{m}$  as functions of  $h$  relative the optimal bandwidth  $h^* = (0.03, 0.04)^\top$ , compare (33) and (34).*

	Min.	Max.	Mean	Median	Stdd.	Skewn.	Kurt.
$\hat{\beta}_1$	-1.541	-0.462	-1.221	-1.260	0.206	1.101	4.082
$\hat{\beta}_2$	-0.075	0.106	0.001	0.002	0.034	0.046	2.717
$\hat{\beta}_3$	-0.144	0.116	0.002	-0.001	0.025	0.108	5.175

Table 3: *Summary statistics on  $(\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3)^\top$  from Section 3.2.*

	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$
$\hat{\beta}_1$	1	0.241	0.368
$\hat{\beta}_2$		1	-0.003
$\hat{\beta}_3$			1

Table 4: *Contemporaneous correlation matrix of  $\hat{\beta}_l$ ,  $l = 1, \dots, 3$ .*

Coefficient	Test Statistic	# of lags
$\hat{\beta}_1$	-2.68	3
$\hat{\beta}_2$	-3.20	1
$\hat{\beta}_3$	-6.11	2

Table 5: *ADF tests on  $\hat{\beta}_1$  to  $\hat{\beta}_3$  for the full IVS model, intercept included in each case. Third column gives the number of lags included in the ADF regression. For the choice of lag length, we started with four lags, and subsequently deleted lag terms, until the last lag term became significant at least at a 5% level. MacKinnon critical values for rejecting the hypothesis of a unit root are -2.87 at 5% significance level, and -3.44 at 1% significance level.*

Dependent variable	Equation		
	$\hat{\beta}_{1,i}$	$\hat{\beta}_{2,i}$	$\hat{\beta}_{3,i}$
$\hat{\beta}_{1,i-1}$	0.978 [24.40]	-0.009 [-1.21]	0.047 [ 3.70]
$\hat{\beta}_{1,i-2}$	0.004 [ 0.08]	0.012 [ 1.63]	-0.047 [-3.68]
$\hat{\beta}_{2,i-1}$	0.182 [ 0.92]	0.861 [ 23.88]	0.134 [ 2.13]
$\hat{\beta}_{2,i-2}$	-0.129 [-0.65]	0.109 [ 3.03]	-0.126 [-2.01]
$\hat{\beta}_{3,i-1}$	0.115 [ 0.97]	-0.019 [-0.89]	0.614 [ 16.16]
$\hat{\beta}_{3,i-2}$	-0.231 [-1.96]	0.030 [ 1.40]	0.248 [ 6.60]
$\bar{R}^2$	0.957	0.948	0.705
$F$ -statistic	2405.273	1945.451	258.165

Table 6: *Estimation results of an VAR(2) of the factor loadings  $\hat{\beta}_i$ .  $t$ -statistics given in brackets,  $\bar{R}^2$  denotes the adjusted coefficient of determination. The estimation includes an intercept and two dummy variables (both not shown), which assume the value one right at those days and one day after, when the corresponding IV observations of the minimum time to maturity string (10 days to expiry) were to be excluded from the estimation of the semiparametric factor model.*

## List of Figures

- 1    *Left panel: call and put implied volatilities observed on 2nd May, 2000. Right panel: data design on 2nd May, 2000, ODAX. . . . .* 44
- 2    *Nadaraya-Watson estimator and semi-parametric factor model fit for 20000502. Bandwidths for both estimates  $h_1 = 0.03$  for the moneyness and  $h_2 = 0.04$  for the time to maturity dimension. . . . .* 45
- 3    *IVS ticks on April 4, 2000, derived from future prices that are interest rate discounted only. Put implied volatility are circles, call implied volatility crosses.* 46
- 4    *IVS ticks on April 4, 2000, derived from future prices that are interest rate discounted and corrected with the implied difference dividend. Put implied volatility are circles, call implied volatility crosses. . . . .* 47
- 5    *Left panel: convergence in the IVS models. Solid line shows the  $L^1$ , the dotted line the  $L^2$  measure of convergence. Total number of iterations is 25. Right panel: average density  $\hat{p}(u) = I^{-1} \sum_{i=1}^I \hat{p}_i(u)$ . Bandwidths are  $h_1 = 0.03$  for moneyness and  $h_2 = 0.04$  for time to maturity. . . . .* 48
- 6    *Factor  $\hat{m}_1$  in the left panel (moneyness lower left axis). Right panel shows contour plots of this function (moneyness left axis). Lines are thick for positive level values, thin for negative ones. The gray scale becomes increasingly lighter the higher the level in absolute value. Stepwidth between contour lines is 0.028, estimated from ODAX data 199801-200105. . . . .* 49
- 7    *Factor  $\hat{m}_2$  in the left panel (moneyness lower left axis). Right panel shows contour plots of this function (moneyness left axis). Lines are thick for positive level values, thin for negative ones. The gray scale becomes increasingly lighter the higher the level in absolute value. Stepwidth between contour lines is 0.225, estimated from ODAX data 199801-200105. . . . .* 50

8	<i>Factor <math>\hat{m}_3</math> in the left panel (moneyness lower left axis). Right panel shows contour plots of this function (moneyness left axis). Lines are thick for positive level values, thin for negative ones. The gray scale becomes increasingly lighter the higher the level in absolute value. Stepwidth between contour lines is 0.240, estimated from ODAX data 199801-200105. . . . .</i>	51
9	<i>Nadaraya-Watson estimator with <math>h = (0.06, 0.25)^\top</math> and semi-parametric factor model with <math>h = (0.03, 0.04)^\top</math> for 20000502. . . . .</i>	52
10	<i>Bias comparison of the Nadaraya-Watson estimator with <math>h = (0.06, 0.25)^\top</math> (left panel) and the semi-parametric factor model with <math>h = (0.03, 0.04)^\top</math> (right panel) for the 17 days to expiry data (black bullets) on 20000502. . . . .</i>	53
11	<i>Bias comparison of the Nadaraya-Watson estimator with <math>h = (0.06, 0.25)^\top</math> (left panel) and the semi-parametric factor model with <math>h = (0.03, 0.04)^\top</math> (right panel) for the 45 days to expiry data (black bullets) on 20000502. . . . .</i>	54
12	<i>Bias comparison of the Nadaraya-Watson estimator with <math>h = (0.06, 0.25)^\top</math> (left panel) and the semi-parametric factor model with <math>h = (0.03, 0.04)^\top</math> (right panel) for the 80 days to expiry data (black bullets) on 20000502. . . . .</i>	55
13	<i>Bias comparison of the Nadaraya-Watson estimator with <math>h = (0.06, 0.25)^\top</math> (left panel) and the semi-parametric factor model with <math>h = (0.03, 0.04)^\top</math> (right panel) for the 136 days to expiry data (black bullets) on 20000502. . . . .</i>	56
14	<i>Upper panel: time series of weights <math>(\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3)^\top</math>. Lower panel: autocorrelation functions. . . . .</i>	57

IVS Ticks 20000502

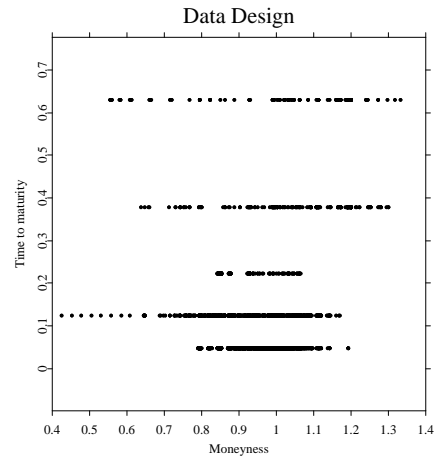
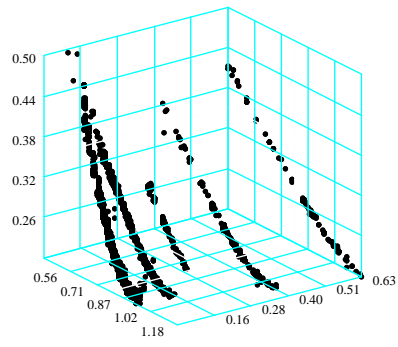
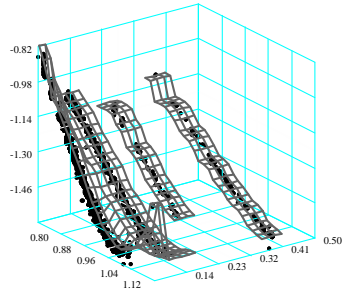


Figure 1: *Left panel: call and put implied volatilities observed on 2nd May, 2000. Right panel: data design on 2nd May, 2000, ODAX.*



Model fit 20000502



Semiparametric factor model fit 20000502

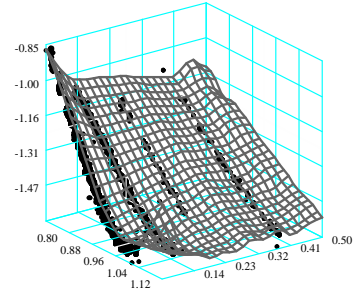


Figure 2: *Nadaraya-Watson estimator and semi-parametric factor model fit for 20000502. Bandwidths for both estimates  $h_1 = 0.03$  for the moneyness and  $h_2 = 0.04$  for the time to maturity dimension.*

### Implied Volatility Surface Ticks

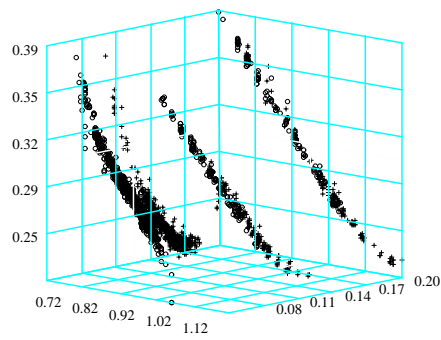


Figure 3: *IVS ticks on April 4, 2000, derived from future prices that are interest rate discounted only. Put implied volatility are circles, call implied volatility crosses.*

### Implied Volatility Surface Ticks

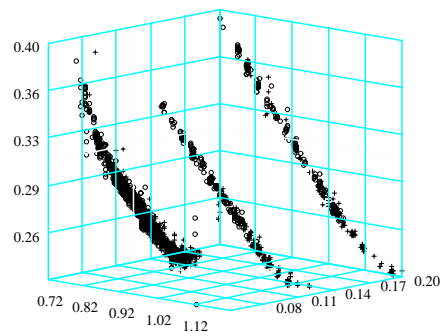


Figure 4: *IVS ticks on April 4, 2000, derived from future prices that are interest rate discounted and corrected with the implied difference dividend. Put implied volatility are circles, call implied volatility crosses.*

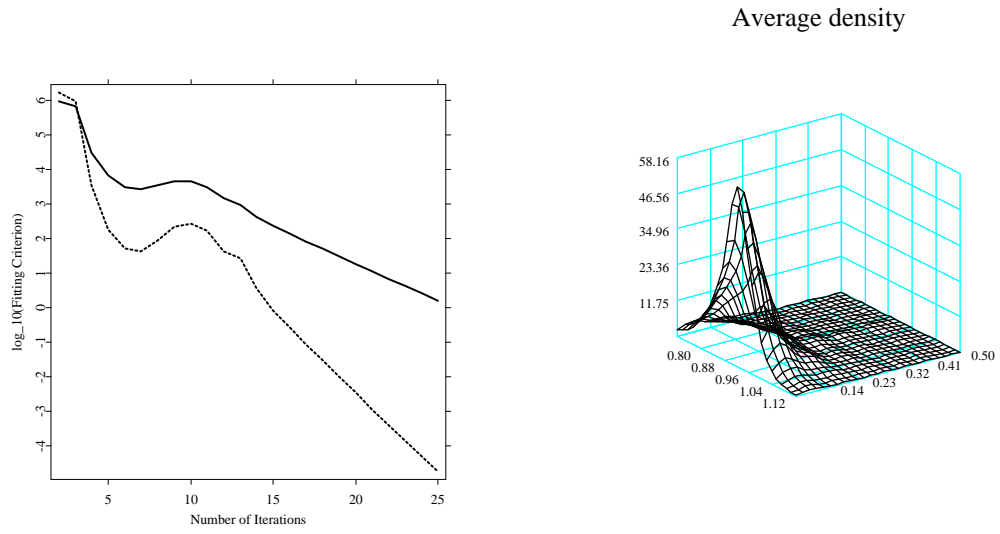


Figure 5: *Left panel: convergence in the IVS models. Solid line shows the  $L^1$ , the dotted line the  $L^2$  measure of convergence. Total number of iterations is 25. Right panel: average density  $\hat{p}(u) = I^{-1} \sum_{i=1}^I \hat{p}_i(u)$ . Bandwidths are  $h_1 = 0.03$  for moneyness and  $h_2 = 0.04$  for time to maturity.*

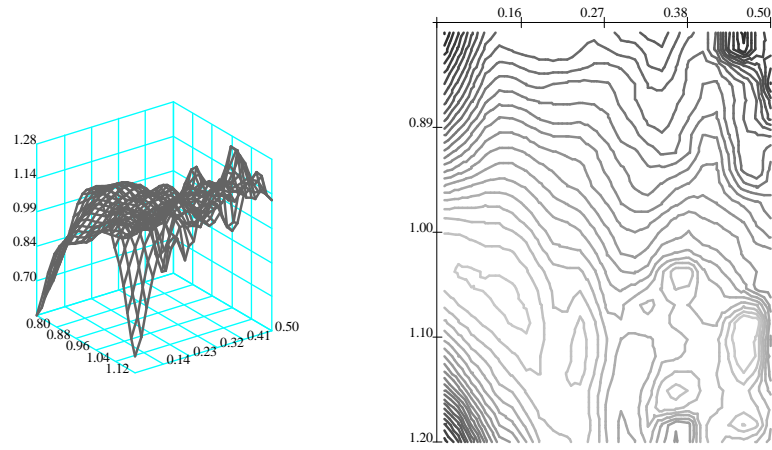


Figure 6: Factor  $\hat{m}_1$  in the left panel (moneyiness lower left axis). Right panel shows contour plots of this function (moneyiness left axis). Lines are thick for positive level values, thin for negative ones. The gray scale becomes increasingly lighter the higher the level in absolute value. Stepwidth between contour lines is 0.028, estimated from ODAX data 199801-200105.

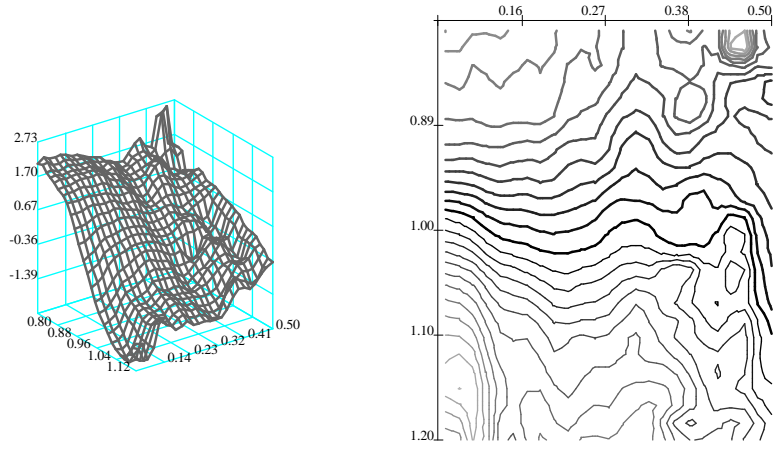


Figure 7: Factor  $\hat{m}_2$  in the left panel (moneyiness lower left axis). Right panel shows contour plots of this function (moneyiness left axis). Lines are thick for positive level values, thin for negative ones. The gray scale becomes increasingly lighter the higher the level in absolute value. Stepwidth between contour lines is 0.225, estimated from ODAX data 199801-200105.

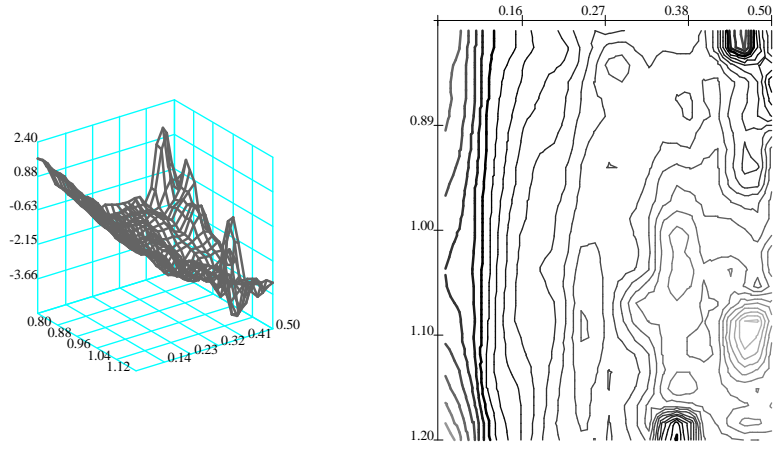
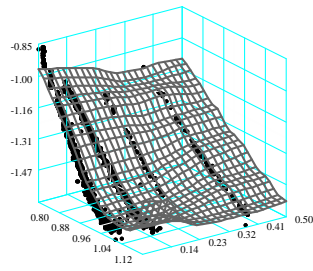


Figure 8: Factor  $\hat{m}_3$  in the left panel (moneyness lower left axis). Right panel shows contour plots of this function (moneyness left axis). Lines are thick for positive level values, thin for negative ones. The gray scale becomes increasingly lighter the higher the level in absolute value. Stepwidth between contour lines is 0.240, estimated from ODAX data 199801-200105.

Model fit 20000502



Semiparametric factor model fit 20000502

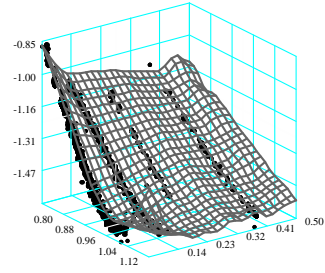
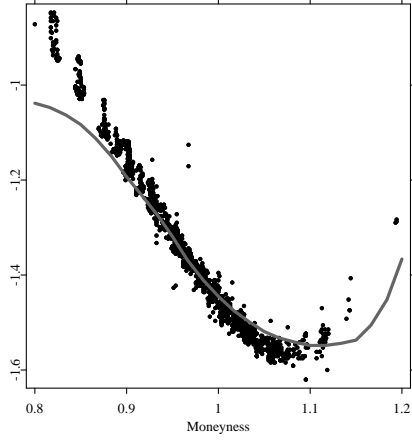


Figure 9: *Nadaraya-Watson estimator with  $h = (0.06, 0.25)^\top$  and semi-parametric factor model with  $h = (0.03, 0.04)^\top$  for 20000502.*



Traditional string fit 20000502, 17 days to exp.



Individual string fit 20000502, 17 days to exp.

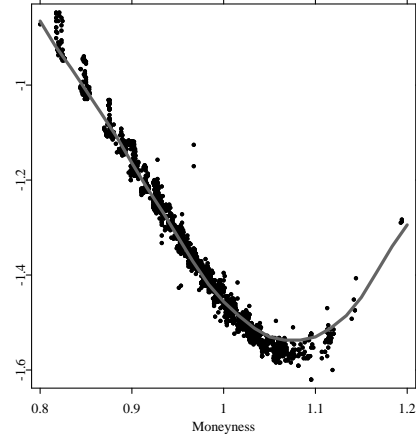
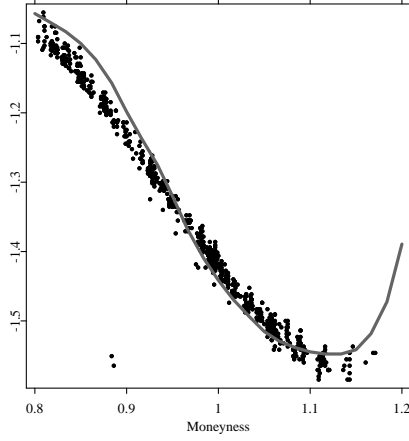


Figure 10: *Bias comparison of the Nadaraya-Watson estimator with  $h = (0.06, 0.25)^\top$  (left panel) and the semi-parametric factor model with  $h = (0.03, 0.04)^\top$  (right panel) for the 17 days to expiry data (black bullets) on 20000502.*

Traditional string fit 20000502, 45 days to exp.



Individual string fit 20000502, 45 days to exp.

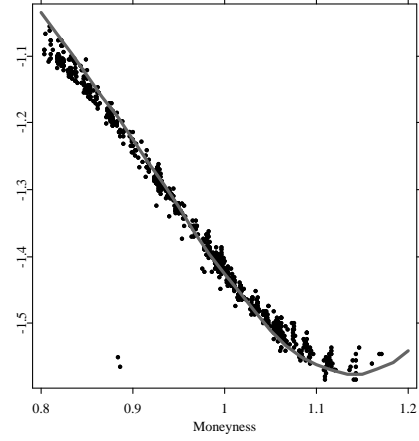
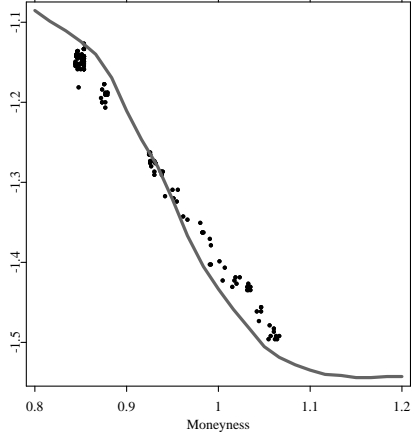


Figure 11: *Bias comparison of the Nadaraya-Watson estimator with  $h = (0.06, 0.25)^\top$  (left panel) and the semi-parametric factor model with  $h = (0.03, 0.04)^\top$  (right panel) for the 45 days to expiry data (black bullets) on 20000502.*

Traditional string fit 20000502, 80 days to exp.



Individual string fit 20000502, 80 days to exp.

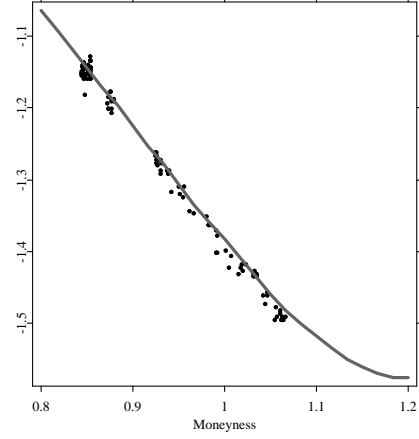
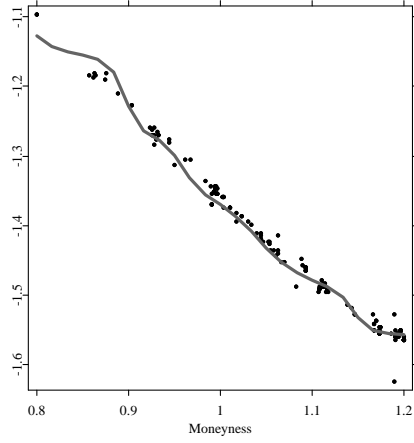


Figure 12: *Bias comparison of the Nadaraya-Watson estimator with  $h = (0.06, 0.25)^\top$  (left panel) and the semi-parametric factor model with  $h = (0.03, 0.04)^\top$  (right panel) for the 80 days to expiry data (black bullets) on 20000502.*

Traditional string fit 20000502, 136 days to exp.



Individual string fit 20000502, 136 days to exp.

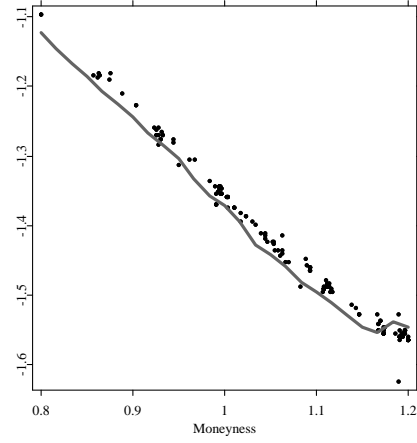


Figure 13: *Bias comparison of the Nadaraya-Watson estimator with  $h = (0.06, 0.25)^\top$  (left panel) and the semi-parametric factor model with  $h = (0.03, 0.04)^\top$  (right panel) for the 136 days to expiry data (black bullets) on 20000502.*

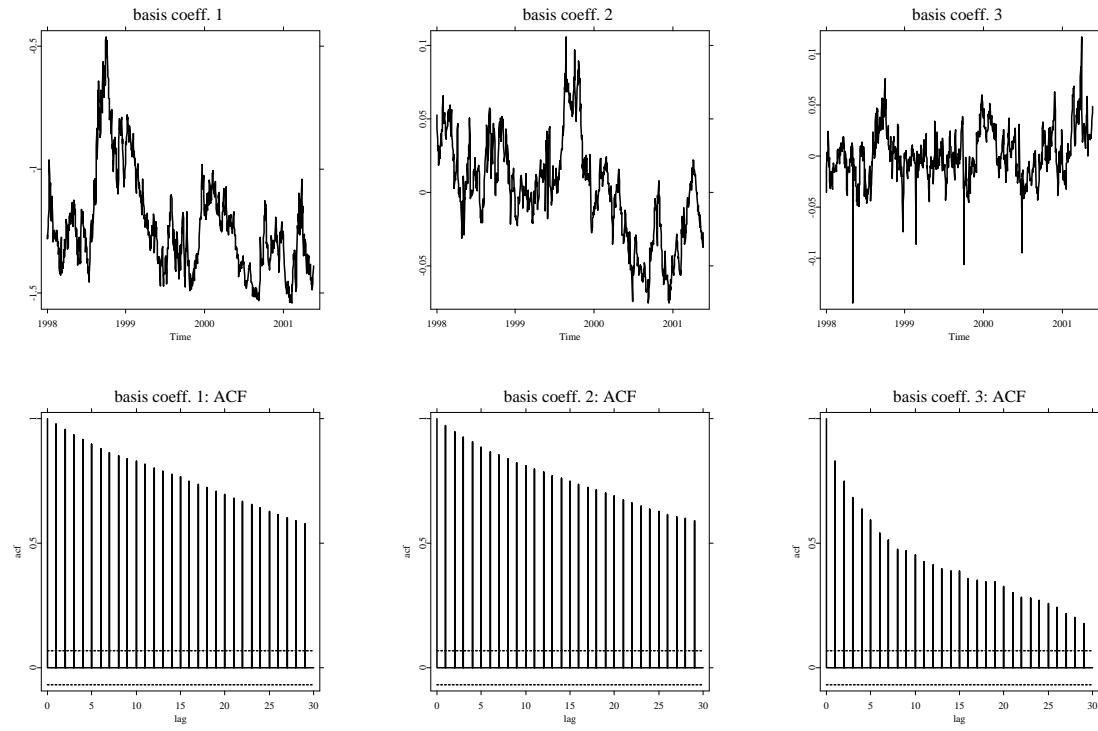


Figure 14: *Upper panel: time series of weights  $(\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3)^\top$ . Lower panel: autocorrelation functions.*